Calculate
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$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-1)!!}{2^n(2n-1)(n+1)!}$, where $(2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$.

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Note that $(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} x^n$. Since $\binom{1/2}{n} = \frac{1}{n!}\prod_{k=1}^{n}(1/2 - k + 1) =$
$\frac{(-1)^{n-1}(2n-3)!!}{2^n n!} = \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)n!}$ then $(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)n!} x^n$

Hence, $\frac{2}{3}\left((1+x)^{3/2} - 1\right) = \int_0^x (1+t)^{1/2} dt = x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)(n+1)!} x^{n+1} \Leftrightarrow$

$\frac{2}{3x}\left((1+x)^{3/2} - 1\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)(n+1)!} x^n$.

Therefore, $\frac{2\left((1+x^4)^{3/2} - 1\right)}{3x^4} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)(n+1)!} x^{4n} \Leftrightarrow$

$\frac{2\left((1+x^4)^{3/2} - 1\right)}{3x^3} = x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n(2n-1)(n+1)!} x^{4n+1}$ implies

$\left(\frac{2\left((1+x^4)^{3/2} - 1\right)}{3x^3}\right)' = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n+1)(2n-1)!!}{2^n(2n-1)(n+1)!} x^{4n} \Leftrightarrow$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n+1)(2n-1)!!}{2^n(2n-1)(n+1)!} x^{4n} = \frac{2\left(2x^4\sqrt{x^4+1} - (x^4+1)\sqrt{x^4+1} + 1\right)}{x^4} - 1$

Then for $x = 1$ we obtain $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n+1)(2n-1)!!}{2^n(2n-1)(n+1)!} = 1$.